Classical & Quantum Waves

WAVES

- Waves are everywhere!
  - water waves
  - seismic waves (earthquakes)
  - acoustic waves

  } in medium (mech. waves)

  - e-m waves (microwaves, radio, visible light, etc.)
  - gravitational waves (black holes, neutron stars)

  } in vacuum or medium

  - quantum mechanics } different...

- 2 types of waves
  - standing (e.g., guitar string)
  - traveling (e.g., ripple on pond)
    - transverse (e.g., e-m)
    - longitudinal (e.g., sound)

- Waves in medium: ~~Mechanical waves~~: Disturbance travels, but <u>not</u> the medium
  - Waves transmit <u>energy</u>!

<u>Traveling waves</u>

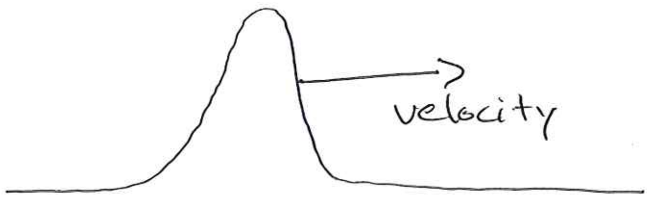
→ Longitudinal waves: disturbance occurs along direction of travel

→ Transverse waves: disturbance occurs $\perp$ to direction of travel

- Both are solutions to the "wave equation"

Characteristics of traveling waves

- Launch a pulse on a long rope

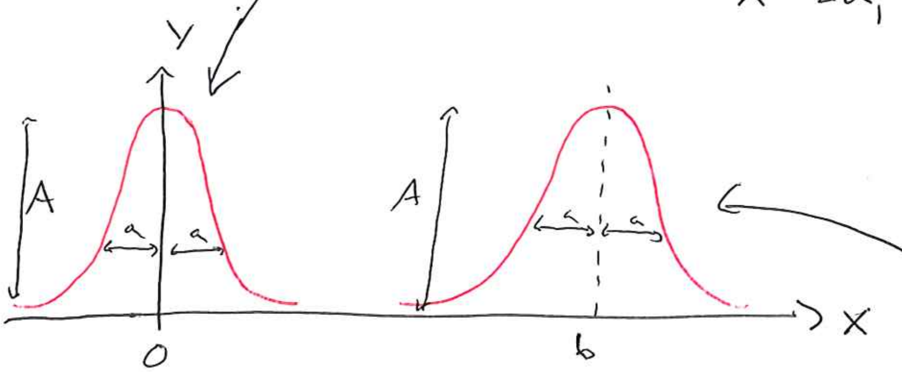


- Model this disturbance as a gaussian ~~pulse~~ function

$$\rightarrow y = Ae^{-x^2/a^2}$$

- when $x=0$, $y=A$

  $x=\pm a$, $y = A/e$



- Can translate function by b to get $y = e^{-\frac{(x-b)^2}{a^2}}$

  → same shape, just translated by b

- change variables to let $b = vt$ (velocity × time)

$$\rightarrow y(x,t) = Ae^{-\frac{(x-vt)^2}{a^2}}$$

↳ Describes traveling wave pulse on rope that moves at a constant rate v while maintaining its shape

- General form of wave going in +x direction:

$$\boxed{f(x,t) = f(x - vt)}$$

- Characteristics
  - wave retains shape as it travels
  - For fixed time t_0, $f(x, t_0)$ represents a snapshot in time of the wave in space
  - For fixed location x_0, wave shape can be determined by observing displacement at location x_0 as function of time

    $f(x_0, t)$

→ Waves link spatial coordinate with time!

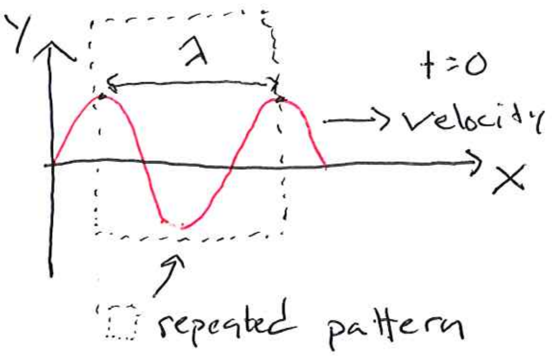
- General form of wave motion:

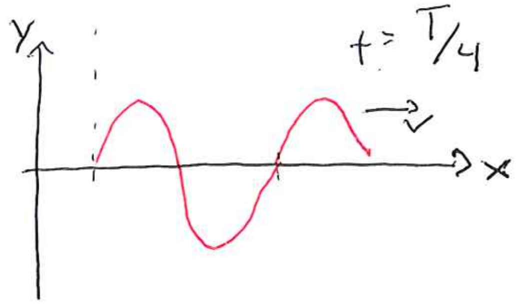
$$\boxed{y(x,t) = \underbrace{f(x-vt)}_{+x \text{ direction}} + \underbrace{g(x+vt)}_{-x \text{ direction}}}$$

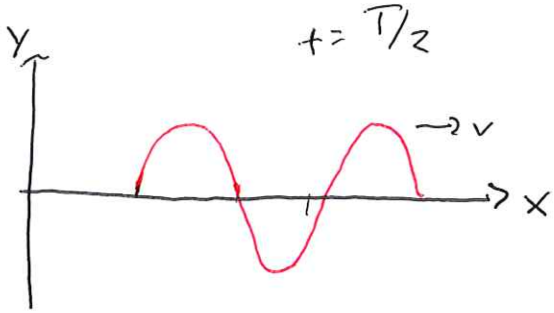
Traveling sinusoidal waves

· VERY IMPORTANT! B/c *any* wave shape can be decomposed into a combination of sinusoidal waves

(→ Fourier Series, Ch. 7)







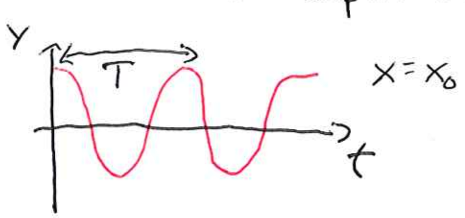
· Wave traveling along x, displacement in ⊥ y-direction → <u>transverse</u> wave

· Displacements lie in a single plane
→ linearly polarized wave

· Spatial pattern repeats
→ repeat distance called "wavelength" λ

· At different snapshots in time (as on left), can see how wave propagates in space

· Alternatively, focus on a single point in space (say $x = x_0$) and watch how the amplitude goes up and down periodically in time:



T: period

- Every point in space acts as a harmonic oscillator!

· Waves link periodic motion in <u>space</u> w/ periodic motion in <u>time</u> [see old timey demo]

· Mathematically express sinusoidal waves as:

$$y(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

where velocity $\boxed{v = \nu \lambda}$ ← frequency, wavelength

↳ Freq, wavelength linked by wave velocity v (e-m waves $v = c$)

$$\boxed{\nu = \frac{1}{T}}$$ → T period

· Can think of frequency as # times per second the crest of the wave passes a given point

· Can express $y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$ in multiple ways:

· $y(x,t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t\right)$ → $\frac{v}{\lambda} = \nu$

$$\rightarrow A \sin\left(\frac{2\pi}{\lambda}x - 2\pi\nu t\right)$$

define $\omega = 2\pi\nu$ angular frequency

$$\boxed{k = \frac{2\pi}{\lambda}}$$ "wave vector" ("spatial" frequency) a.k.a. "wave number"

$$\Rightarrow \boxed{y(x,t) = A \sin(kx - \omega t)}$$

(most common form)

↳ complex form

$$\boxed{y(x,t) = A e^{i(kx - \omega t)}}$$

also $v = \frac{\omega}{k} = \frac{2\pi\nu\lambda}{2\pi} = \nu\lambda$ ✓

Wave equation

- Recall that general form for any wave motion is:

$$y = f(x-vt) + g(x+vt)$$

$\rightarrow$ general solution to the "wave eq." which we derive below

- Let's simplify math by changing variables to $u \equiv x - vt$

$\rightarrow f(x-vt) \equiv F(u)$

- Let's write out its derivatives

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right] = \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

since $u \equiv x - vt$, $\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial u} 1\right] = \frac{\partial^2 f}{\partial u^2}(1) + \frac{\partial f}{\partial u}(0)$$

using $\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial F}{\partial u}(1) \Rightarrow \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial u^2} \Rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}}$$

· Do the same procedure as above for

$\frac{\partial f}{\partial t}$, $\frac{\partial^2 f}{\partial t^2}$ $\Rightarrow$ $\boxed{\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2}}$

$\left[\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} \right]$

$\Rightarrow$ combine these two eqs.

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

$\left[y = f(x-vt) + g(x+vt) \right]$

· Do the same thing for $g(x+vt)$

$$\Rightarrow \frac{\partial^2 g}{\partial t^2} = v^2 \frac{\partial^2 g}{\partial x^2}$$

· Total result : $\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}}$ 1D wave eq.

w/ solutions $y = f(x-vt) + g(x+vt)$

→ Describes any wave propagation in 1D, not just displacement. y could represent air pressure, or temperature, voltage in transmission line etc.

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

w/ $\psi = f(x-vt) + g(x+vt)$

ψ : physical quantity (general form)